Free convection laminar boundary layers in oscillatory flow

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The effect of harmonic oscillations in the magnitude of the surface temperature on the free-convection laminar velocity and temperature boundary layers on a flat plate is analysed. Low- and high-frequency solutions are developed separately. The results obtained are in striking contrast to the corresponding results for forced-convection flows.

1. Introduction

The study of laminar boundary layers in oscillatory flow with a steady mean was initiated by Lighthill (1954) who considered the effects of fluctuations in stream velocity on the skin friction and heat transfer for plates and cylinders. Stuart (1955) in an attempt to verify certain results of Lighthill's analysis, discussed the problem of flow over an infinite flat plate with suction, when the main stream oscillates in time about a constant mean, obtaining an exact solution of the Navier–Stokes equations. Lighthill in his paper had studied two particular cases of flow over a semi-infinite flat plate (Blasius layer) and of flow near a stagnation point (Hiemenz layer). Hill (1958) has studied the effects of free-stream oscillations on laminar boundary layers of Howarth (1938) flow. He has obtained three different solutions for low-, intermediate- and high-frequency ranges. For low- and high-frequency oscillations he has followed the method of Lighthill, but oscillations in the intermediate-frequency range required a different treatment. Further contributions on the subject have been made by Carrier & DiPrima (1957), Nickerson (1958), Rosenzweig (1959), Rott & Rosenzweig (1960), Glauert (1956) and Watson (1959). The importance of the phenomenon of response of an otherwise steady laminar boundary layer to small disturbances need hardly be stressed.

The present paper is devoted to a study of free-convection laminar boundary-layer flows from a vertical flat plate, when the plate temperature oscillates in time about a constant non-zero mean, while the free stream is isothermal. The treatment is restricted to small-amplitude oscillations only. This enables us to effect linearization of the equations. Two different solutions for low- and high-frequency ranges are developed. It is found that in the low-frequency range, the oscillating component of skin friction always lags behind the plate temperature oscillations while the rate of heat transfer has a phase lead. In the high-frequency range, the velocity and temperature in the boundary layer are of the 'shear wave' type, predicting a phase lead of 45° in the rate of heat-transfer fluctuations and an equivalent phase lag in the skin-friction oscillations.

2. Basic equations

Consider a heated vertical flat plate whose temperature oscillates about a non-zero mean, while the free-stream temperature is constant. The boundary-layer equations for an incompressible fluid are

$$\frac{\partial \bar{u}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = g\beta(\bar{T}-\bar{T}_\infty)+\nu\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \tag{2.1}$$

$$\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{2.2}$$

$$\frac{\partial \bar{T}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \alpha\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}, \tag{2.3}$$

where g is the acceleration due to gravity, β is the coefficient of volume expansion, α is the thermal diffusivity, and $\bar{T}_\infty$ is the temperature of the free stream.

Introducing dimensionless quantities

$$\left.\begin{aligned} x = \bar{x}/L, \quad y = \bar{y}/L, \quad t = \nu\bar{t}/L^2, \\ u = \bar{u}L/\nu, \quad v = \bar{v}L/\nu, \quad G = (\bar{T}-\bar{T}_\infty)/(\bar{T}_W-\bar{T}_\infty), \end{aligned}\right\} \tag{2.4}$$

where L is the characteristic length, i.e. $[g\beta(\bar{T}_W-\bar{T}_\infty)/\nu^2]^{-\frac{1}{3}}$, equations (2.1), (2.2) and (2.3) become respectively

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}+G, \tag{2.5}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{2.6}$$

and

$$\frac{\partial G}{\partial t}+u\frac{\partial G}{\partial x}+v\frac{\partial G}{\partial y} = \frac{1}{\sigma}\frac{\partial^2 G}{\partial y^2}, \tag{2.7}$$

where σ is the Prandtl number and $\bar{T}_W$ is the constant mean temperature of the plate.

The boundary conditions to be satisfied are

$$\left.\begin{aligned} y = 0: \quad u = 0, \quad v = 0, \quad G = (1+\epsilon\cos\omega t), \quad \epsilon \ll 1; \\ y \to \infty: \quad u = 0, \quad G = 0, \end{aligned}\right\} \tag{2.8}$$

where ω is the dimensionless frequency $\omega L^2/\nu$.

3. Method of solution

In solving the above differential equations it is convenient to adopt the complex notation for harmonic functions. The solutions will be obtained in terms of complex functions, the real parts of which will have physical significance. The plate temperature, which can be written as $[\bar{T}_W+\epsilon(\bar{T}_W-\bar{T}_\infty)e^{i\omega t}]$, consists of a basic steady distribution $\bar{T}_W$ with a superimposed weak time-varying distribution $\epsilon(\bar{T}_W-\bar{T}_\infty)e^{i\omega t}$.

We now write u, v and G as the sum of steady and small oscillating components:

$$\left.\begin{aligned} u &= u_s+\epsilon u_1 e^{i\omega t}, \\ v &= v_s+\epsilon v_1 e^{i\omega t}, \\ G &= G_s+\epsilon G_1 e^{i\omega t}, \end{aligned}\right\} \tag{3.1}$$

where u_s, v_s, G_s is the steady mean flow and satisfies

$$\left.\begin{aligned} u_s\frac{\partial u_s}{\partial x}+v_s\frac{\partial u_s}{\partial y} &= \frac{\partial^2 u_s}{\partial y^2}+G_s,\\ \frac{\partial u_s}{\partial x}+\frac{\partial v_s}{\partial y} &= 0,\\ u_s\frac{\partial G_s}{\partial x}+v_s\frac{\partial G_s}{\partial y} &= \frac{1}{\sigma}\frac{\partial^2 G_s}{\partial y^2},\end{aligned}\right\} \tag{3.2}$$

with the boundary conditions

$$\left.\begin{aligned} y=0:&\quad u_s=v_s=0,\quad G_s=1;\\ y\to\infty:&\quad u_s\to 0,\qquad\quad G_s\to 0.\end{aligned}\right\} \tag{3.3}$$

Neglecting squares of ϵ and dividing by $e^{i\omega t}$, we find that u_1, v_1, G_1 satisfy the following differential set,

$$\left.\begin{aligned} i\omega u_1+u_1\frac{\partial u_s}{\partial x}+u_s\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_s}{\partial y}+v_s\frac{\partial u_1}{\partial y} &= \frac{\partial^2 u_1}{\partial y^2}+G_1,\\ \frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y} &= 0,\\ i\omega G_1+u_1\frac{\partial G_s}{\partial x}+u_s\frac{\partial G_1}{\partial x}+v_1\frac{\partial G_s}{\partial y}+v_s\frac{\partial G_1}{\partial y} &= \frac{1}{\sigma}\frac{\partial^2 G_1}{\partial y^2},\end{aligned}\right\} \tag{3.4}$$

with the boundary conditions

$$\left.\begin{aligned} y=0:&\quad u_1=v_1=0,\quad G_1=1;\\ y\to\infty:&\quad u_1\to 0,\qquad\quad G_1\to 0.\end{aligned}\right\} \tag{3.5}$$

Equations (3.2) and (3.3) are the well-known boundary-layer equations which describe the steady-state free-convection flow past a vertical flat plate. These equations have been integrated numerically by various workers and their solution is rather well known. We shall now solve the differential set (3.4) subject to the boundary conditions (3.5). As mentioned in the introduction two separate solutions will be obtained, one for small frequencies and the other for high frequencies.

Low-frequency fluctuations

It is convenient to write u_1, v_1, and G_1 as the sum of in-phase and out-of-phase components. We substitute

$$u_1 = u_r+iu_2,\quad v_1 = v_r+iv_2,\quad G_1 = G_r+iG_2, \tag{3.6}$$

in (3.4) and separate real and imaginary parts to get

$$-\omega u_2+u_r\frac{\partial u_s}{\partial x}+u_s\frac{\partial u_r}{\partial x}+v_r\frac{\partial u_s}{\partial y}+v_s\frac{\partial u_r}{\partial y} = \frac{\partial^2 u_r}{\partial y^2}+G_r, \tag{3.7}$$

$$\frac{\partial u_r}{\partial x}+\frac{\partial v_r}{\partial y} = 0, \tag{3.8}$$

$$-\omega G_2+u_r\frac{\partial G_s}{\partial x}+u_s\frac{\partial G_r}{\partial x}+v_r\frac{\partial G_s}{\partial y}+v_s\frac{\partial G_r}{\partial y} = \frac{1}{\sigma}\frac{\partial^2 G_r}{\partial y^2}, \tag{3.9}$$

with the boundary conditions

$$\left.\begin{aligned} y = 0: \quad & u_r = v_r = 0, \quad G_r = 1; \\ y \to \infty: \quad & u_r \to 0, \qquad\quad G_r \to 0, \end{aligned}\right\} \tag{3.10}$$

and

$$\omega u_r + u_2 \frac{\partial u_s}{\partial x} + u_s \frac{\partial u_2}{\partial x} + v_2 \frac{\partial u_s}{\partial y} + v_s \frac{\partial u_2}{\partial y} = \frac{\partial^2 u_2}{\partial y^2} + G_2, \tag{3.11}$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0. \tag{3.12}$$

$$\omega G_r + u_2 \frac{\partial G_s}{\partial x} + u_s \frac{\partial G_2}{\partial x} + v_2 \frac{\partial G_s}{\partial y} + v_s \frac{\partial G_2}{\partial y} = \frac{1}{\sigma} \frac{\partial^2 G_2}{\partial y^2}, \tag{3.13}$$

with the boundary conditions

$$\left.\begin{aligned} y = 0: \quad & u_2 = v_2 = G_2 = 0; \\ y \to \infty: \quad & u_2 \to 0, \quad G_2 \to 0. \end{aligned}\right\} \tag{3.14}$$

The difference in phase between the longitudinal velocity and the temperature fluctuations at a point within the boundary layer and in the plate temperature fluctuations is $\alpha_1 = \tan^{-1}(u_2/u_r)$ and $\alpha_2 = \tan^{-1}(G_2/G_r)$. When the frequency of oscillation is low, it is to be expected that the phase shift will be small. Therefore one would expect u_2 and G_2 to be small relative to u_r and G_r. Thus when ω is small, the terms $(-\omega u_2)$ and $(-\omega G_2)$ can be neglected in (3.7) and (3.9). u_r, v_r and G_r will then be the quasi-steady solution corresponding to $\omega = 0$. This can be seen from the fact that the same equations can be obtained by substituting $u = u_s + u_r$, $v = v_s + v_r$, $G = G_s + G_r$ in the steady-flow boundary-layer equations. Thus, we easily find that

$$u_r = \theta_0(\partial u_s/\partial \theta_0) \quad v_r = \theta_0(\partial v_s/\partial \theta_0), \quad G_r = \theta_0(\partial G_s/\partial \theta_0), \tag{3.15}$$

where $\theta_0 = (\overline{T}_W - \overline{T}_\infty)$. It now remains to determine u_2, v_2, and G_2. It is evident that the solution of the basic steady flow must be known beforehand. Squire (1953) has given a simple solution of the basic steady flow using the Karman–Pohlhausen method. We shall use the same method to solve (3.11), (3.12) and (3.13). Accordingly we assume the following expressions for u_2 and G_2,

$$u_2 = B_1(\eta - 3\eta^3 + 2\eta^4), \tag{3.16}$$

$$G_2 = A_1(\eta - 3\eta^3 + 2\eta^4) + \tfrac{1}{2}\sigma\omega\delta^2(\eta^2 - 2\eta^3 + \eta^4), \tag{3.17}$$

where $\eta = y/\delta$, δ being the dimensionless boundary-layer thickness. A_1 and B_1 are functions of x and are to be determined. The approximate expressions for u_s and G_s as given by Squire are

$$u_s = u_x \eta (1 - \eta)^2, \tag{3.18}$$

$$G_s = (1 - \eta)^2, \tag{3.19}$$

where

$$u_x = 5{\cdot}17\,[x/(\sigma + \tfrac{20}{21})]^{\frac{1}{2}}, \tag{3.20}$$

$$\delta = 3{\cdot}93\,[(\sigma + \tfrac{20}{21})\,x/\sigma^2]^{\frac{1}{4}}. \tag{3.21}$$

Integrating (3.11) and (3.13) from $y = 0$ to $y = \delta$ and using (3.12) and (3.14) we obtain the averaging conditions as

$$\omega \int_0^1 u_r\, d\eta + \frac{2}{\delta} \frac{\partial}{\partial x}\left(\delta \int_0^1 u_2 u_s\, d\eta\right) + \frac{1}{\delta^2}\left(\frac{\partial u_2}{\partial \eta}\right)_{\eta=0} - \int_0^1 G_2\, d\eta = 0, \tag{3.22}$$

$$\omega \int_0^1 G_r\, d\eta + \frac{1}{\delta} \frac{\partial}{\partial x}\left\{\delta \int_0^1 (y_2 G_s + u_s G_2)\, d\eta\right\} + \frac{1}{\sigma\delta^2}\left(\frac{\partial G_2}{\partial \eta}\right)_{\eta=0} = 0. \tag{3.23}$$

From these we determine the values of A_1 and B_1 as

$$A_1 = \frac{-\sqrt{15}\omega N(132N^2+216\sigma+953)}{297N^2+112\,(14+3\sigma)}x^{\frac{1}{2}}, \quad B_1 = \frac{-40\omega(15N^2+14)}{297N^2+112(14+3\sigma)}x \tag{3.24}$$

where $$N = (\sigma+\tfrac{20}{21})^{\frac{1}{2}}.$$

High-frequency oscillations

For high frequencies, Lighthill has shown that the oscillatory flow is to a close approximation an ordinary 'shear wave' unaffected by the mean flow. The flow field can be described as a superposition of the steady mean flow and a 'shear wave' flow corresponding to the oscillating component of the plate temperature. The thickness of the steady boundary layer is large compared to the oscillating boundary-layer thickness which is of the order $(\nu/\omega)^{\frac{1}{2}}$ and one can visualize the entire oscillating boundary layer as being contained within that region of the steady boundary layer wherein the non-linear inertia terms are negligible. Therefore, if the frequency is high enough, the differential set (3.4) reduces to

$$i\omega u_1 = \frac{\partial^2 u_1}{\partial y^2}+G_1, \quad i\omega G_1 = \frac{1}{\sigma}\frac{\partial^2 G_1}{\partial y^2}, \tag{3.25}$$

from which we easily obtain

$$u_1 = [1/i\omega(\sigma-1)]\,[\exp\{-(i\omega)^{\frac{1}{2}}y\} - \exp\{-(i\omega\sigma)^{\frac{1}{2}}y\}], \tag{3.26}$$

$$G_1 = \exp\{-(i\omega\sigma)^{\frac{1}{2}}y\}. \tag{3.27}$$

4. Discussion of the results

When the frequency of the oscillation is small, the longitudinal component of velocity and temperature may be written in the form

$$u = u_s+\epsilon R_1\cos(\omega t+\alpha_1), \tag{4.1}$$

$$G = G_s+\epsilon R_2\cos(\omega t+\alpha_2), \tag{4.2}$$

where

$$R_1 = (u_r^2+u_2^2)^{\frac{1}{2}}, \quad R_2 = (G_r^2+G_2^2)^{\frac{1}{2}}, \quad \alpha_1 = \tan^{-1}(u_2/u_r), \quad \alpha_2 = \tan^{-1}(G_2/G_r).$$

The velocity and temperature in 'shear-wave' flow are

$$u = u_s+\{\epsilon R_3/\omega(1-\sigma)\}\cos(\omega t-\alpha_3), \tag{4.3}$$

$$G = G_s+\epsilon R_4\cos(\omega t-\alpha_4), \tag{4.4}$$

where
$$R_3 = (P^2+Q^2)^{\frac{1}{2}}, \quad R_4 = \exp\{-(\tfrac{1}{2}\sigma\omega)^{\frac{1}{2}}y\};$$

$$\alpha_3 = \tan^{-1}(Q/P), \quad \alpha_4 = (\tfrac{1}{2}\sigma\omega)^{\frac{1}{2}}y;$$

$$P = \exp\{-(\tfrac{1}{2}\omega)^{\frac{1}{2}}y\}\sin\{(\tfrac{1}{2}\omega)^{\frac{1}{2}}y\}-\exp\{-(\tfrac{1}{2}\omega\sigma)^{\frac{1}{2}}y\}\sin\{(\tfrac{1}{2}\omega\sigma)^{\frac{1}{2}}y\}$$

$$Q = \exp\{-(\tfrac{1}{2}\omega\sigma)^{\frac{1}{2}}y\}\cos\{(\tfrac{1}{2}\omega\sigma)^{\frac{1}{2}}y\}-\exp\{-(\tfrac{1}{2}\omega)^{\frac{1}{2}}y\}\cos\{(\tfrac{1}{2}\omega)^{\frac{1}{2}}y\}.$$

The functions u_r, u_2, G_r, G_2 are exhibited in figure 1 for $\sigma = 0{\cdot}73$. Since B_1 is negative, u_2 is always negative but u_r is positive near the plate and negative near the edge of the boundary layer so that the phase angle is negative near the plate. Near the edge of the boundary layer the velocity fluctuations have a phase lead over the plate-temperature oscillations. On the other hand, α_2 is always positive. The amplitude and phase angle of the velocity and temperature profile in 'shear-wave' flow are exhibited in figures 2 and 3.

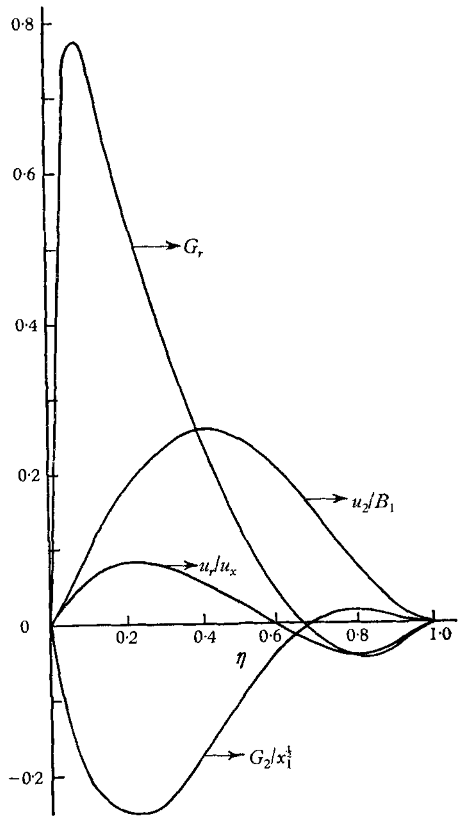

FIGURE 1. Function graph.

Local heat transfer

The local heat transfer from the surface to the fluid may be calculated using Fourier's law $q = -k(\partial \overline{T}/\partial \overline{y})_{\overline{y}=0}$. Introducing dimensionless variables from (2.4) we have

$$q = -\frac{k\nu^2}{g\beta L^4}\left[\left(\frac{\partial G_s}{\partial y}\right)_{y=0} + \epsilon\, e^{i\omega t}\left(\frac{\partial G_1}{\partial y}\right)_{y=0}\right]. \tag{4.5}$$

The temperature gradient in 'shear-wave' flow is given by

$$\mathrm{Re}\,[e^{i\omega t}(\partial G_1/\partial y)_{y=0}] = -(\omega\sigma)^{\frac{1}{2}}\cos(\omega t + \tfrac{1}{4}\pi). \tag{4.6}$$

Its amplitude increases with frequency and its phase is ahead of that of the fluctuations of the surface temperature by 45°. On the other hand, the wall velocity gradient in 'shear-wave' flow is given by

$$\mathrm{Re}[e^{i\omega t}(\partial u_1/\partial y)_{y=0}] = \{\epsilon/\omega^{\frac{1}{2}}(\sigma^{\frac{1}{2}}+1)\}\cos(\omega t - \tfrac{1}{4}\pi), \tag{4.7}$$

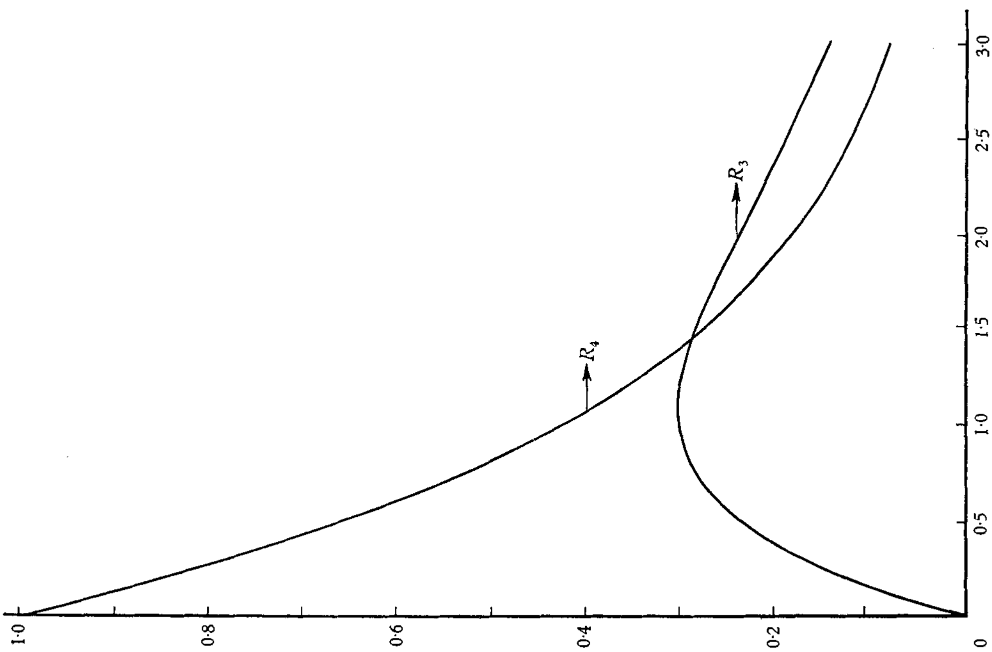

FIGURE 2. 'Shear-wave' solution (amplitude).

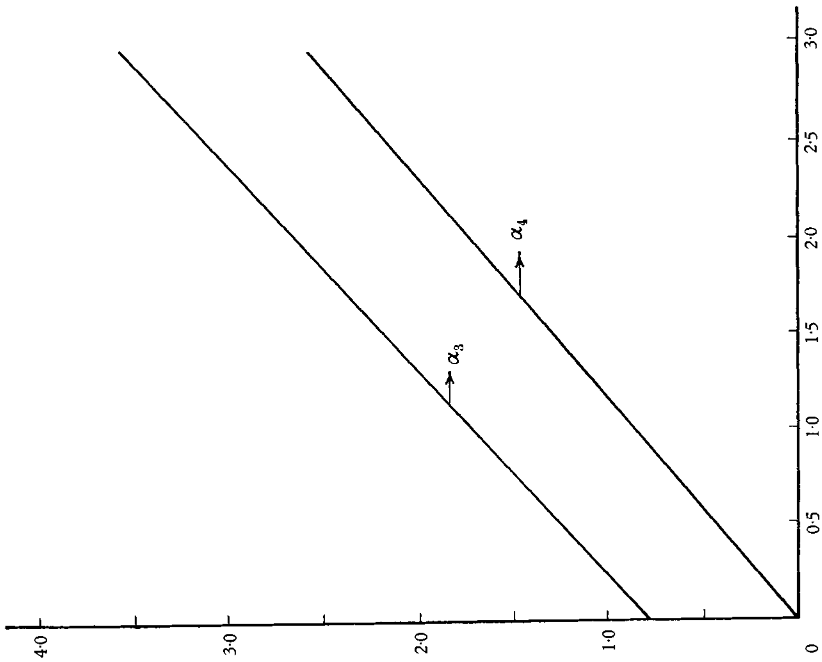

FIGURE 3. 'Shear-wave' solution (phase angle).

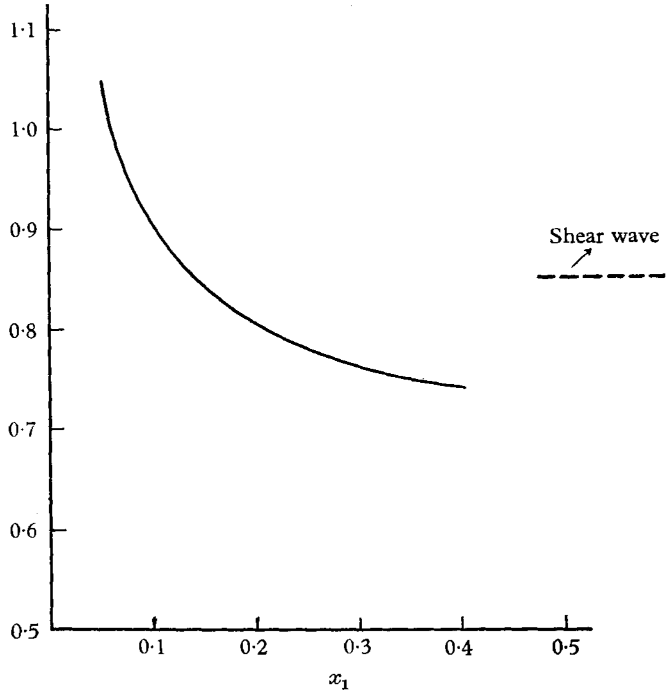

FIGURE 4. Amplitude of oscillating wall temperature gradient.

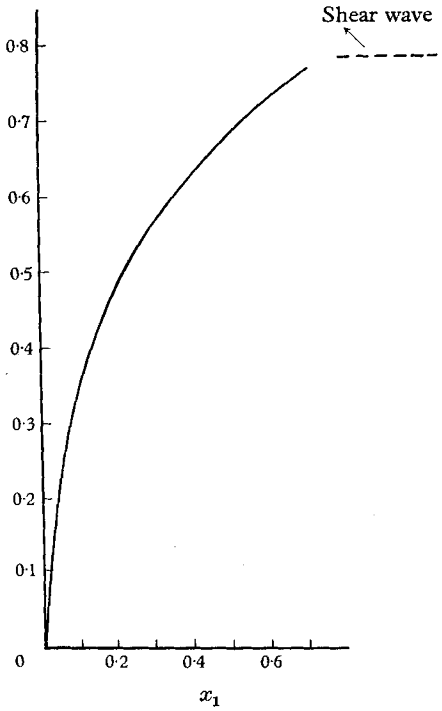

FIGURE 5. Phase angle of oscillating wall temperature gradient.

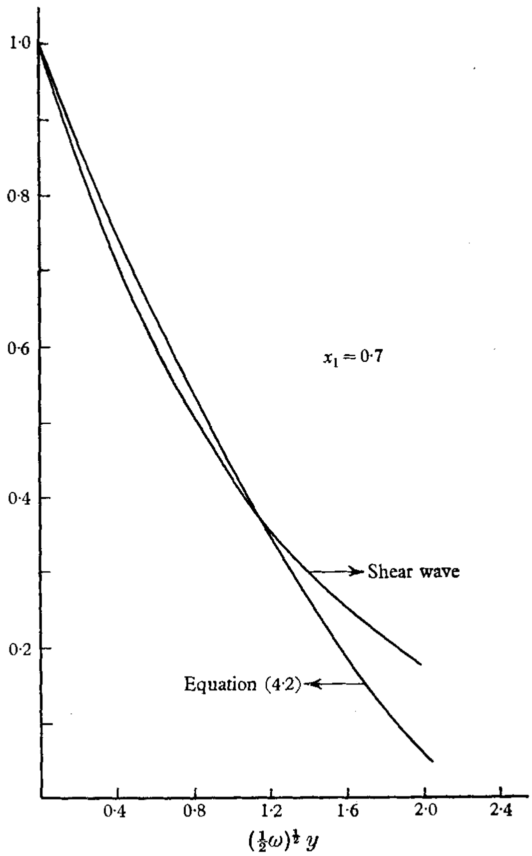

FIGURE 6. Amplitude of temperature oscillations.

the amplitude of which decreases with frequency and its phase lags behind the plate temperature oscillations by 45°. These results are in striking contrast to the case of forced-convection flow. For low-frequency oscillations we have

$$\mathrm{Re}[e^{i\omega t}(\partial G_1/\partial y)_{y=0}] = -\{(\omega\sigma)^{\frac{1}{2}} x_1^{-\frac{1}{4}}/3{\cdot}93N^{\frac{1}{2}}\}(\tfrac{25}{4}+A_1^2)^{\frac{1}{2}}\cos(\omega t+\psi), \qquad (4.8)$$

where $$\psi = \tan^{-1}(-\tfrac{2}{5}A_1), \quad x_1 = x\omega^2.$$

The variations of amplitude and phase angle of the wall temperature gradient as a function of x_1 are shown in figures 4 and 5. The corresponding asymptotic values are also shown. It may be observed that the phase angle approaches its asymptotic value when $x_1 \approx 0{\cdot}7$. However, the amplitude attains its asymptotic value much more rapidly. The low- and high-frequency solutions may be matched on the basis of heat-transfer oscillations, taking the matching point as the frequency at which the low-frequency solution predicts a phase advance equal to that of the shear-wave solution. Thus $x_1 \approx 0{\cdot}7$ may be taken as the boundary between the regions of applicability of high- and low-frequency solutions. It is of interest to compare the temperature profiles obtained on the basis of low and high frequencies. The comparison is made in figure 6 for a value of $x_1 = 0{\cdot}7$.

Overall heat transfer

The overall heat transfer Q is also a quantity of engineering interest. It is given by

$$Q = b\int_0^x q\,dx = \frac{bk\nu^2 x^{\frac{3}{4}}}{g\beta L^4(3{\cdot}93)[\sigma^{-1}(1+20/21\sigma)]^{\frac{1}{4}}}[\tfrac{8}{3}+\epsilon(\tfrac{100}{9}+\tfrac{16}{25}A_1^2)^{\frac{1}{2}}\cos(\omega t+\lambda)],$$

where $$\lambda = \tan^{-1}(-\tfrac{6}{25}A_1).$$

Since A_1 is always negative, the oscillating component of the overall heat transfer has a phase lead over the surface temperature oscillations.

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